MATH 110 – SOLUTIONS TO THE FIRST PRACTICE MIDTERM

PEYAM TABRIZIAN

(1) (a) Write out the definition of a subspace

Definition: A subset W of a vector space V is called a **subspace** of V if the following three conditions are satisfied:

(1) The zero vector $\mathbf{0}$ of V is also in W

(2) W is closed under addition, that is: If $\mathbf{w_1}$ and $\mathbf{w_2}$ are in W, then $\mathbf{w_1} + \mathbf{w_2}$ is in W

(3) W is closed under scalar multiplication, that is: If $\mathbf{w}$ is in W and c is in $\mathbb{F}$, then $c\mathbf{w}$ is in W

(b) Write out the definition of direct sum

Definition: Let V be a vector space and $U_1, \cdots, U_n$ be subspaces of V. Then we say that $V = U_1 \oplus \cdots \oplus U_n$ if every v in V can be uniquely written as a sum of vectors in $U_1, \cdots U_n$. This means:

(A) Every v in V can be written as $v = \mathbf{u_1} + \mathbf{u_2} + \cdots + \mathbf{u_n}$, where each $\mathbf{u_i}$ is in U_i with $i = 1, \cdots, n$ (that is, there *exists* such a representation)[1]

(B) If $\mathbf{v} = \mathbf{u_1} + \cdots + \mathbf{u_n}$ **AND** $\mathbf{v} = \mathbf{w_1} + \cdots + \mathbf{w_n}$, where each $\mathbf{u_i}$ and $\mathbf{w_i}$ is in U_i for $i = 1, \cdots, n$, then for every such i, we have $\mathbf{u_i} = \mathbf{w_i}$ (that is, such a representation is *unique*)

(c) Suppose that U and W are subspaces of a vector space V such that $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. Show that $V = U \oplus W$.

Based on our definition in (b), we need to show that every vector $\mathbf{v}$ in V can be uniquely written as $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u}$ is in U, and $\mathbf{w}$ is in W.

Let $\mathbf{v}$ be an arbitrary vector in V. We need to check that (A) and (B) in (b) are true.

(A) First of all, by *definition* of $V = U + W$, we know that there exist vectors $\mathbf{u}$ in U and $\mathbf{w}$ in W such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. Hence (A) is satisfied.

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[1]It's ok to say $V = U_1 + \cdots + U_n$, but then you would have to define what the sum of n vector spaces are

(B) Suppose you can write $\mathbf{v} = \mathbf{u} + \mathbf{w}$ and $\mathbf{v} = \mathbf{u}' + \mathbf{w}'$, where $\mathbf{u}, \mathbf{u}'$ are in U and $\mathbf{w}, \mathbf{w}'$ are in W. You want to show that $\mathbf{u}' = \mathbf{u}$ and $\mathbf{w}' = \mathbf{w}$ [2]

Then we get:

$$\begin{aligned}
\mathbf{u} + \mathbf{w} &= \mathbf{u}' + \mathbf{w}' && \text{by the above two equations} \\
-\mathbf{u}' + (\mathbf{u} + \mathbf{w}) &= -\mathbf{u}' + (\mathbf{u}' + \mathbf{w}') && \text{adding } -\mathbf{u}' \text{ on the left} \\
(-\mathbf{u}' + \mathbf{u}) + \mathbf{w} &= (-\mathbf{u}' + \mathbf{u}') + \mathbf{w}' && \text{by associativity} \\
(-\mathbf{u}' + \mathbf{u}) + \mathbf{w} &= \mathbf{0} + \mathbf{w}' && \text{by definition of } -\mathbf{u}' \\
(-\mathbf{u}' + \mathbf{u}) + \mathbf{w} &= \mathbf{w}' && \text{by definition of } \mathbf{0} \\
((-\mathbf{u}' + \mathbf{u}) + \mathbf{w}) + (-\mathbf{w}) &= \mathbf{w}' + (-\mathbf{w}) && \text{adding } -\mathbf{w} \text{ on the right} \\
(-\mathbf{u}' + \mathbf{u}) + (\mathbf{w} + (-\mathbf{w})) &= \mathbf{w}' - \mathbf{w} && \text{by associativity} \\
(-\mathbf{u}' + \mathbf{u}) + \mathbf{0} &= \mathbf{w}' - \mathbf{w} && \text{by definition of } -\mathbf{w} \\
-\mathbf{u}' + \mathbf{u} &= \mathbf{w}' - \mathbf{w} && \text{(by definition of } \mathbf{0})
\end{aligned}$$

However, since $\mathbf{u}$ and $\mathbf{u}'$ are in U and U is a subspace of V, it is closed under scalar multiplication, and hence $-\mathbf{u}'$ is in U, and since U is closed under addition, $-\mathbf{u}' + \mathbf{u}$ is in U

Similarly, since $\mathbf{w}$ and $\mathbf{w}'$ are in W and W is a subspace of V, it is closed under scalar multiplication, and hence $-\mathbf{w}$ is in W, and since W is closed under addition, $\mathbf{w}' - \mathbf{w}$ is in W

It follows that $-\mathbf{u}' + \mathbf{u}$ is in U, but also $-\mathbf{u}' + \mathbf{u} = \mathbf{w}' - \mathbf{w}$ is in W, hence $-\mathbf{u}' + \mathbf{u}$ is in $U \cap W = \{\mathbf{0}\}$ by assumption, therefore $-\mathbf{u}' + \mathbf{u} = \mathbf{0}$ and $\mathbf{w}' - \mathbf{w} = -\mathbf{u}' + \mathbf{u} = \mathbf{0}$.

But then, we get:

$$\begin{aligned}
-\mathbf{u}' + \mathbf{u} &= \mathbf{0} \\
\mathbf{u}' + (-\mathbf{u}' + \mathbf{u}) &= \mathbf{u}' + \mathbf{0} && \text{adding } \mathbf{u}' \text{ to the left} \\
(\mathbf{u}' - \mathbf{u}') + \mathbf{u} &= \mathbf{u}' && \text{by associatitivy and definition of } \mathbf{0} \\
\mathbf{0} + \mathbf{u} &= \mathbf{u}' && \text{by definition of } -\mathbf{u}' \\
\mathbf{u} &= \mathbf{u}' && \text{by definition of } \mathbf{0}
\end{aligned}$$

Hence $\boxed{\mathbf{u}' = \mathbf{u}}$
And:

[2] You *could* have used Prop 1.8 and show that if $\mathbf{u} + \mathbf{w} = \mathbf{0}$ (where $\mathbf{u}$ is in U and $\mathbf{w}$ is in W), **BUT** then you would **HAVE** to prove that Prop 1.8 is true (that it is equivalent with your definition in (b))

$$\begin{aligned}
\mathbf{w}' - \mathbf{w} &= \mathbf{0} \\
(\mathbf{w}' - \mathbf{w}) + \mathbf{w} &= \mathbf{0} + \mathbf{w} && \text{adding } \mathbf{w} \text{ on the right} \\
\mathbf{w}' + (-\mathbf{w} + \mathbf{w}) &= \mathbf{w} && \text{by associativity and the definition of } \mathbf{0} \\
\mathbf{w}' + \mathbf{0} &= \mathbf{w} && \text{by definition of } -\mathbf{w} \\
\mathbf{w}' &= \mathbf{w} && \text{by definition of } \mathbf{0}
\end{aligned}$$

Hence $\boxed{\mathbf{w}' = \mathbf{w}}$, and (B) is proved!

Therefore $V = U \oplus W$. □

(2) Prove that if $(\mathbf{v_1}, \cdots, \mathbf{v_n})$ spans V, then so does the list:

$$(\mathbf{v_1} - \mathbf{v_2}, \cdots, \mathbf{v_{n-1}} - \mathbf{v_n}, \mathbf{v_n})$$

Let $\mathbf{v}$ be an arbitrary vector in V.

<u>Goal:</u> We want to find scalars $a_1, a_2, \cdots, a_n$ in $\mathbb{F}$ such that:

(1) $$\mathbf{v} = a_1 (\mathbf{v_1} - \mathbf{v_2}) + \cdots + a_{n-1} (\mathbf{v_{n-1}} - \mathbf{v_n}) + a_n \mathbf{v_n}$$

<u>What we know:</u> Since $(\mathbf{v_1}, \cdots, \mathbf{v_n})$ spans V, we know that there exist scalars $b_1, b_2, \cdots, b_n$ in $\mathbb{F}$ such that:

(2) $$v = b_1 \mathbf{v_1} + b_2 \mathbf{v_2} + \cdots + b_n \mathbf{v_n}$$

<u>Scratchwork:</u>[3]
Expanding (1) out, we get:

$$\begin{aligned}
\mathbf{v} &= a_1 (\mathbf{v_1} - \mathbf{v_2}) + \cdots + a_{n-1} (\mathbf{v_{n-1}} - \mathbf{v_n}) + a_n \mathbf{v_n} \\
&= a_1 \mathbf{v_1} - a_1 \mathbf{v_2} + a_2 \mathbf{v_2} - a_3 \mathbf{v_3} + \cdots + a_{n-1} \mathbf{v_{n-1}} - a_{n-1} \mathbf{v_n} + a_n \mathbf{v_n} && \text{by distributivity} \\
&= a_1 \mathbf{v_1} + (a_2 - a_1) \mathbf{v_2} + (a_3 - a_2) \mathbf{v_3} + \cdots + (a_n - a_{n-1}) \mathbf{v_n} && \text{by distributivity}
\end{aligned}$$

But comparing this with $\mathbf{v} = b_1 \mathbf{v_1} + b_2 \mathbf{v_2} + \cdots + b_n \mathbf{v_n}$, we **guess** [4] that:

(3) $$a_1 = b_1$$
(4) $$a_2 - a_1 = b_2$$
(5) $$\cdots$$
(6) $$a_n - a_{n-1} = b_n$$

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[3] Although this is scratchwork, I would prefer if you wrote this on the exam, because in this way I would know how you obtained your $a_1, \cdots, a_n$

[4] **CAREFUL**, this is just a *guess*, it doesn't necessarily *imply* that $a_1 = b_1$, etc., for example, the list $(\mathbf{v_1}, \cdots, \mathbf{v_n})$ could be linearly dependent, then there are many different guesses that are correct

That is:

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= a_1 + b_2 \\ &\cdots \\ a_n &= a_{n-1} + b_n \end{aligned}$$

Therefore:

$$\begin{aligned} a_1 &= b_1 \\ a_2 &= b_1 + b_2 \\ &\cdots \\ a_n &= b_1 + \cdots + b_n \end{aligned}$$

In other words: $\boxed{a_i = b_1 + b_2 + \cdots + b_i}$ $(i = 1, \cdots n)$ [5], [6]

<u>Proof that your guess works:</u>[7]:

Let's calculate:

$$\begin{aligned} & a_1 (\mathbf{v_1} - \mathbf{v_2}) + a_2 (\mathbf{v_2} - \mathbf{v_3}) + \cdots + a_{n-1} (\mathbf{v_{n-1}} - \mathbf{v_n}) + a_n \mathbf{v_n} \\ = & a_1 \mathbf{v_1} - a_1 \mathbf{v_2} + a_2 \mathbf{v_2} - a_2 \mathbf{v_3} + \cdots + a_{n-1} \mathbf{v_{n-1}} - a_{n-1} \mathbf{v_n} + a_n \mathbf{v_n} && \text{by distributivity} \\ = & a_1 \mathbf{v_1} + (a_2 - a_1) \mathbf{v_2} + (a_3 - a_2) \mathbf{v_3} + \cdots + (a_n - a_{n-1}) \mathbf{v_n} && \text{by distributivity again} \\ = & b_1 \mathbf{v_1} + b_2 \mathbf{v_2} + \cdots + b_n \mathbf{v_n} && \text{by (3)} \\ = & \mathbf{v} && \text{by (2)} \end{aligned}$$

Therefore, working backwards:

$$\mathbf{v} = a_1 (\mathbf{v_1} - \mathbf{v_2}) + a_2 (\mathbf{v_2} - \mathbf{v_3}) + \cdots + a_{n-1} (\mathbf{v_{n-1}} - \mathbf{v_n}) + a_n \mathbf{v_n}$$

Hence (1) is satisfied with our choices of $a_1, \cdots, a_n$!

Since $\mathbf{v}$ was arbitrary, we get $\text{Span}(\mathbf{v_1} - \mathbf{v_2}, \cdots, \mathbf{v_{n-1}} - \mathbf{v_n}, \mathbf{v_n}) = V$. □

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[5] Technically, you should prove this by induction, just as in the solutions to HW2

[6] It is **CRUCIAL** that you give us an *explicit* formula of the a_i (what you want to show) in terms of b_i (what you know). Just writing $a_i - a_{i-1} = b_i$ would be **WRONG** because you're not guaranteed that this recurrence relation has a solution!

[7] You **HAVE** to include this, the above was just scratchwork, i.e. a recipe to obtain your a_i

(3) (a) Write out the definition of linear independence

Definition: Let V be a vector space and $(\mathbf{v_1}, \cdots, \mathbf{v_n})$ a list of vectors in V. Then $(\mathbf{v_1}, \cdots, \mathbf{v_n})$ is **linearly independent** in V if for any scalars $a_1, \cdots, a_n$ in $\mathbb{F}$:

$$a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n} = \mathbf{0} \text{ implies } a_1 = a_2 = \cdots = a_n = 0$$

(b) Suppose that $p_0, p_1, \cdots, p_m$ are polynomials in $P_m(\mathbb{F})$ such that $p_j(2) = 0$ for each j, Prove that $(p_0, \cdots, p_m)$ is not linearly independent in $P_m(\mathbb{F})$.

Let's prove this by **contradiction**[8].

Suppose that $(p_0, \cdots, p_m)$ is linearly independent in $P_m(\mathbb{F})$.

Then $(p_0, \cdots, p_m)$ is a linearly independent list of exactly $m + 1$ vectors in P_m. However, since dim$(P_m(\mathbb{F}))$ is exactly $m + 1$ [9], we conclude that in fact $(p_0, \cdots, p_m)$ is a **basis** for $P_m(\mathbb{F})$ [10].

In particular, by definition of a basis, we get that Span $(p_0, \cdots, p_m) = P_m(\mathbb{F})$.

But then, since Span $(p_0, \cdots, p_m) = P_m(\mathbb{F})$ and $p(x) = 1$ (the constant-1 polynomial) is in $P_m(\mathbb{F})$, by definition of Span, get that there are constants $a_0, \cdots, a_m$ such that:

$$p = a_0p_0 + \cdots + a_mp_m$$

Hence, by definition of a polynomial (as a function), we get that for **all** x:

$$p(x) = a_0p_0(x) + \cdots + a_mp_m(x)$$

In particular, setting $x = 2$, we get $p(2) = 1$ (by definition of p) and $p_j(2) = 0$ ($j = 0, 1, \cdots, m$), by assumption, hence the above equation becomes:

$$1 = a_0(0) + \cdots + a_0(0)$$

That is:

$$1 = 0$$

Which is a contradiction $\Rightarrow\Leftarrow$. Hence $(p_0, \cdots, p_m)$ cannot be linearly independent in $P_m(\mathbb{F})$, hence it is linearly dependent in $P_m(\mathbb{F})$ □

[8] Write that on the exam!

[9] You can take this for granted, or prove it by writing finding a basis of P_m and proving that it is a basis

[10] This is Prop 2.17 on page 32. For cases like this, you don't have to reprove all of Prop 2.17, it's enough just to write directly that $(p_0, \cdots, p_m)$ is a basis for $P_m(\mathbb{F})$ as long as you say *why* you can apply it in this situation